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p-Adic holomorphy rings and Kochen rings

Cristina Maria Marques*, Otto Endler†

Departamento de Matemática, Universidade Federal de Minas Gerais, Caixa Postal 702, 30.161-970, Belo Horizonte, MG, Brazil

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Abstract

The p-adic analogue of the real holomorphy ring is defined. With the introduction of the Kochen ring we obtain properties of the p-adic holomorphy ring which are similar to certain properties of the real holomorphy ring, proved by Schülting (1979).

0. Introduction

This paper deals with the p-adic analogue of the real holomorphy ring $H_{\mathbb{R}}(K)$, defined as the intersection of the valuation rings belonging to real places $\mathscr{P}: K \to \mathbb{R} \cup \{\infty\}$. The analogy is to be understood in a wide sense, namely that $\mathbb{R}$ is to be replaced not only by the p-adic number field $\mathbb{Q}_p$ but by an arbitrary finite extension E of $\mathbb{Q}_p$. We show that $H_E(K)$ has properties which are similar to certain properties of $H_{\mathbb{R}}(K)$, proved by Schülting [6].

In the case of an *unramified* extension E of $\mathbb{Q}_p$, a proof was given in [1], using the Kochen ring of type $(1, f)$ as defined in [4]. The proof in the general case was obtained later by Roquette [5] who, for this purpose, introduced the Kochen ring $R_E(K)$ of type E. This notion seems to be of interest for its own sake and can be considered as a valuable complement to [4, Section 6].

After studying the relationship between p-valuations a p-adic places in Section 1, we reduce in Section 2 the proof of the properties of $H_E(K)$ to a condition which states that a certain set of p-valuations is "saturated". The fact that this condition is always satisfied turns out to be a consequence of properties of the Kochen ring $R_E(K)$, which are proved in Section 3.

Notations. For any (Krull) valuation v of K, we denote by $\mathcal{O}_v$, $\mathcal{M}_v$, vK, K_v its valuation ring, maximal ideal, value group, residue field, respectively. $\mathcal{O}_{\mathscr{P}}$, $\mathcal{M}_{\mathscr{P}}$ and $K_{\mathscr{P}}$ have similar meanings with respect to a place $\mathscr{P}$ of K.

*Corresponding author.

†Deceased on 05/12/1988.

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For any finite extension E of $\mathbb{Q}_p$, the subscript E in the notations $\mathcal{O}_E$, $\mathcal{M}_E$, etc. refers to the unique prolongation v_E to E of the p-adic valuation v_p of $\mathbb{Q}_p$, whose valuation ring is denoted by $\mathbb{Z}_p$.

1. p-valuations and p-adic places

Let K be a field of characteristic zero. A valuation v of K is called a *p-valuation*, if there are integers $e_v \geq 1$ and $f_v \geq 1$ with the following properties: There exist exactly e_v elements $\gamma \in vK$ such that $0 < \gamma \leq vp$, and f_v is the degree of K_v over its prime field. It is obvious that in this case K_v is a field of p^{f_v} elements and that $vp = e_v \cdot v\pi$, where $v\pi$ is the smallest positive element of vK; moreover, $\mathbb{Z} \cdot v\pi$ is the smallest nontrivial convex subgroup of vK (and is equal to vK if and only if v has rank one).

We recall that, for any convex subgroup Δ of vK, the valuation v of K decomposes in a valuation $\dot{v}$ of K, defined by

$$\dot{v}a = va + \Delta \quad (a \in K \setminus \{0\}),$$

with value group $\dot{v}K = vK/\Delta$ and valuation ring $\mathcal{O}_{\dot{v}} \supseteq \mathcal{O}_v$, and a valuation $\mathring{v}$ of the residue field $K_{\dot{v}}$ of $\dot{v}$, defined by

$$\mathring{v}(a + \mathcal{M}_{\dot{v}}) = va \quad (a \in \mathcal{O}_{\dot{v}}),$$

with value group Δ, valuation ring $\mathcal{O}_v/\mathcal{M}_{\dot{v}}$ and residue field isomorphic to K_v. On the other hand, for any couple of valuations w of K and u of K_w, there is a unique valuation v of K (up to equivalence) such that $w = \dot{v}$ and $u = \mathring{v}$. If Δ is the smallest nontrivial convex subgroup of vK, the $\dot{v}$ and $\mathring{v}$ are called the *coarse* and the *central* valuation, respectively, belonging to v. In particular, if v is a p-valuation of K and $\Delta = \mathbb{Z} \cdot v\pi$, then we have the following proposition.

Proposition 1.1. (a) *$\mathcal{O}_{\dot{v}}$ is equal to the ring of fractions $(\mathcal{O}_v)_M$, where $M = \{p^j \mid j \in \mathbb{N}\}$.*
(b) *$\mathring{v}$ is a p-valuation of rank one with $e_{\mathring{v}} = e_v$ and $f_{\mathring{v}} = f_v$.*

Proof. (a) Since $p^{-1} \in \mathcal{O}_{\dot{v}} \setminus \mathcal{O}_v$ and $\mathcal{O}_{\dot{v}}$ is the smallest subring S of K such that $\mathcal{O}_v \cup \{p^{-1}\} \subseteq S$, we have $\mathcal{O}_{\dot{v}} \subseteq (\mathcal{O}_v)_M$. The equality holds since $\dot{v}(a \cdot p^{-j}) = \dot{v}a \geq 0$ for all $a \in \mathcal{O}_v$ and $j \in \mathbb{N}$.

(b) Since the residue field of $\mathring{v}$ is isomorphic to K_v, we have $f_{\mathring{v}} = f_v$. Since $\mathring{v}(\pi^j + \mathcal{M}_{\dot{v}}) = j \cdot v\pi$ $(1 \leq j \leq e_v)$ are the only positive elements of $\mathring{v}K_{\dot{v}}$ which are not larger than $\mathring{v}p = vp$, we have $e_{\mathring{v}} = e_v$. Moreover, $\mathring{v}K_{\dot{v}} = \mathbb{Z} \cdot v\pi$ has rank one. □

Let E be a finite extension of $\mathbb{Q}_p$. Then v_E is a p-valuation of rank one, and E is a totally ramified (hence Eisenstein) extension of $\mathbb{Q}_{p,f_E}$ of degree e_E, where $\mathbb{Q}_{p,f}$ denotes the unique unramified extension of $\mathbb{Q}_p$ of degree f. Moreover, (E, v_E) is complete, and any valued field (K, v), such that v is a p-valuation of rank one, has a completion of the form (E, v_E) with $e_v = e_E$ and $f_v = f_E$. By part (b) the Proposition 1.1, the same is true for the central valuation $\mathring{v}$ belonging to an arbitrary p-valuation v of K. This fact enables us to define the "type" of a p-valuation as follows.

We say that a p-valuation v of K is *of type E*, if E contains an isomorphic image of the completion of $(K_{\mathring{v}}, \mathring{v})$. In this case, e_v divides e_E and f_v divides f_E, and the canonical homomorphism from $\mathcal{O}_{\mathring{v}}$ onto $K_{\mathring{v}}$ obviously extends to a place $\mathscr{P}: K \to E \cup \{\infty\}$. More precisely we have the following proposition.

Proposition 1.2. *For any p-valuation v of K of type E there exists a place $\mathscr{P}: K \to E \cup \{\infty\}$ such that $\mathcal{O}_{\mathscr{P}} = \mathcal{O}_{\mathring{v}}$.*

We are going to prove that also the converse is true.

Proposition 1.3. *For any place $\mathscr{P}: K \to E \cup \{\infty\}$, where E is a finite extension of $\mathbb{Q}_p$, there exists a p-valuation v of K of type E such that $\mathcal{O}_{\mathring{v}} = \mathcal{O}_{\mathscr{P}}$.*

Proof. We may assume that $K_{\mathscr{P}} \subseteq E$; then $A = \mathcal{O}_E \cap K_{\mathscr{P}}$ is a valuation ring of $K_{\mathscr{P}}$ of rank one, and therefore, $\mathscr{P}^{-1}A$ is a valuation ring of K which is strictly contained in $\mathcal{O}_{\mathscr{P}}$ and is maximal with respect to this property. Let v be a valuation of K with $\mathcal{O}_v = \mathscr{P}^{-1}A$; then vK has a smallest nontrivial convex subgroup Δ, and $\mathcal{O}_{\mathscr{P}}$, A are the valuation rings of $\dot{v}$, $\mathring{v}$, respectively. Obviously, $\mathring{v}$ is a p-valuation of rank one; therefore, $\Delta = \mathbb{Z} \cdot v\pi$ for some $\pi \in \mathcal{M}_v \setminus \mathcal{M}_{\dot{v}}$ such that $\mathring{v}(\mathscr{P}\pi^j) = v\pi^j$ $(j = 1, \ldots, e_v)$ are the only elements $\gamma \in \Delta$ with $0 < \gamma \leq \mathring{v}(\mathscr{P}p) = vp$. They are also the only elements in vK with this property, since Δ is a convex subgroup of vK. Moreover, v and $\mathring{v}$ have isomorphic residue fields; therefore, v is a p-valuation. Since (E, v_E) contains a completion of $(K_{\mathscr{P}}, \mathring{v})$, v is of type E. $\square$

2. The p-adic holomorphy ring $H_E(K)$

The *real* holomorphy ring $H_{\mathbb{R}} = H_{\mathbb{R}}(K)$ of a field K is defined as the intersection $\bigcap_{\mathscr{P} \in \mathscr{S}_{\mathbb{R}}} \mathcal{O}_{\mathscr{P}}$, where $\mathscr{S}_{\mathbb{R}} = \mathscr{S}_{\mathbb{R}}(K)$ denotes the set of all places $\mathscr{P}: K \to \mathbb{R} \cup \{\infty\}$. We recall that $\mathscr{S}_{\mathbb{R}}$ is nonempty if and only if K is *formally real*, that is, K admits an ordering or, equivalently, $-1 \notin \Sigma K^2$. Even in this case we may have $H_{\mathbb{R}} = K$; in fact, this occurs if and only if all orderings of K are archimedean.

Denoting by $\mathcal{A}_{\mathbb{R}} = \mathcal{A}_{\mathbb{R}}(K)$ the set of all valuation rings of K with formally real residue field, Schülting [6] proved that for any formally real field K the following statements hold:

$(1_{\mathbb{R}})$ $H_{\mathbb{R}} = \bigcap_{\mathcal{O} \in \mathcal{A}_{\mathbb{R}}} \mathcal{O}$.

$(2_{\mathbb{R}})$ For any valuation ring $\mathcal{O}$ of K, $H_{\mathbb{R}} \subseteq \mathcal{O}$ implies $\mathcal{O} \in \mathcal{A}_{\mathbb{R}}$.

$(3_{\mathbb{R}})$ $H_{\mathbb{R}} = \mathbb{Q}[\{1/(1+q) \mid q \in \Sigma K^2\}] = \{a \in K \mid r \pm a \in \Sigma K^2$ for some $r \in \mathbb{N}\}$.

$(4_{\mathbb{R}})$ $H_{\mathbb{R}}$ is a Prüfer ring of K.

Similarly, for any finite extension E of $\mathbb{Q}_p$, we introduce the p-adic holomorphy ring of K of type E, and we shall show that it has analogous properties.

We say that K is *formally p-adic of type E*, if K admits at least one p-valuation of type E. By Propositions 1.2 and 1.3, this occurs if and only if K admits at least one nontrivial place $\mathscr{P}: K \to E \cup \{\infty\}$. Let $\mathscr{S}_E = \mathscr{S}_E(K)$ be the set of all places of this type. The intersection $\bigcap_{\mathscr{P} \in \mathscr{S}_E} \mathcal{O}_{\mathscr{P}}$ is called the *p-adic holomorphy ring of K of type E* and will be denoted by $H_E = H_E(K)$. Note that $H_E = K$ if and only if all p-valuations of K of type E have rank one.

Let $\mathscr{A}_E = \mathscr{A}_E(K)$ be the set of all valuation rings of K whose residue field is formally p-adic of type E. From well-known facts about the composition of places (cf. [2, Section 8]) we conclude the following proposition.

Proposition 2.1. *For any valuation ring $\mathcal{O}$ of K we have $\mathcal{O} \in \mathscr{A}_E$ if and only if $\mathcal{O}_{\mathscr{P}} \subseteq \mathcal{O}$ for some $\mathscr{P} \in \mathscr{S}_E$. Therefore $\{\mathcal{O}_{\mathscr{P}} \mid \mathscr{P} \in \mathscr{S}_E\} \subseteq \mathscr{A}_E$ and*

$$(1_E) \qquad H_E = \bigcap_{\mathcal{O} \in \mathscr{A}_E} \mathcal{O}.$$

A nonempty set $\mathscr{A}$ of valuation rings of a field K is called *saturated* if for any valuation ring $\mathcal{O}$ of K, $\bigcap_{\mathcal{O}' \in \mathscr{A}} \mathcal{O}' \subseteq \mathcal{O}$ implies that $\mathcal{O}'' \subseteq \mathcal{O}$ for some $\mathcal{O}'' \in \mathscr{A}$. From Proposition 2.1 it follows that the analogue (2_E) of $(2_{\mathbb{R}})$ holds if and only if $\mathscr{A}_E$ is saturated, and this occurs if and only if $\{\mathcal{O}_{\mathscr{P}} \mid \mathscr{P} \in \mathscr{S}_E\}$ is saturated.

Concerning the analogue (3_E) or $(3_{\mathbb{R}})$, we consider the subrings $(\bigcap_{v \in \vartheta} \mathcal{O}_v)_M \subseteq \mathbb{Q} \cdot \bigcap_{v \in \vartheta} \mathcal{O}_v$ of K, where ϑ is the set $\vartheta_E(K)$ of all p-valuations of K of type E.[1] Actually, these two rings are equal (since any prime number $p' \neq p$ is a unit in $\bigcap_{v \in \vartheta} \mathcal{O}_v$), and from the propositions of Section 1 we conclude that $\mathbb{Q} \cdot \bigcap_{v \in \vartheta} \mathcal{O}_v = (\bigcap_{v \in \vartheta} \mathcal{O}_v)_M \subseteq \bigcap_{v \in \vartheta} (\mathcal{O}_v)_M = \bigcap_{v \in \vartheta} \mathcal{O}_{\hat{v}} = \bigcap_{\mathscr{P} \in \mathscr{S}_E} \mathcal{O}_{\mathscr{P}} = H_E$. Therefore the analogue (3_E) of $(3_{\mathbb{R}})$ is valid if and only if in this inclusion the equality holds.

Finally, the analogue of $(4_{\mathbb{R}})$ can be proved even in the following stronger form:

(4_E) $\quad$ *H_E is a Bezout ring of K,*

i.e., every finitely generated ideal of H_E is principal and K is the field of quotients of H_E. In fact, by the Principal Ideal Theorem (see 4, Section 6]), $\bigcap_{v \in \vartheta} \mathcal{O}_v$ is a Bezout ring of K (since $\# K_v \leq p^{f_E}$ for every $v \in \vartheta$), and so is its over-ring $\bigcap_{v \in \vartheta} \mathcal{O}_{\hat{v}} = H_E$.

In order to complete the analogy with the real case, we have still to prove that $\mathscr{A}_E$ is saturated and that $H_E = \mathbb{Q} \cdot \bigcap_{v \in \vartheta} \mathcal{O}_v$. The proof will be given here under the hypothesis that the set $\{\mathcal{O}_v \mid v \in \vartheta\}$ is saturated. Later, in Section 3, we shall prove that this hypothesis is satisfied for $\vartheta = \vartheta_E(K)$ whenever this set is nonempty.

Theorem 2.2. *Assume that ϑ is saturated and let $R = \bigcap_{v \in \vartheta} \mathcal{O}_v$. Then*

(3_E) $H_E = \mathbb{Q} \cdot R = R_M$, *and*

(2_E) *for any valuation ring $\mathcal{O}$ of K, $H_E \subseteq \mathcal{O}$ implies $\mathcal{O} \in \mathscr{A}_E$.*

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[1] Instead of $\vartheta = \vartheta_E(K)$, we could consider any subset ϑ of $\vartheta_E(E)$ which is sufficiently large in the sense that $\{\mathcal{O}_{\hat{v}} \mid v \in \vartheta\} = \{\mathcal{O}_{\mathscr{P}} \in \mathscr{S}_E\}$.

Proof. Since R is a Bezout ring of K, so is the over-ring $\mathbb{Q} \cdot R$. In particular, $\mathbb{Q} \cdot R$ is integrally closed and is therefore equal to $\bigcap_{\theta \in \mathbb{B}} \theta$, where $\mathbb{B}$ is the set of all valuation rings of K which contain $\mathbb{Q} \cdot R$. For any $\theta \in \mathbb{B}$ we have $R \subseteq \theta$; therefore, by assumption, $\theta_v \subseteq \theta$ for some $v \in \vartheta$. Actually $\theta_v \neq \theta$, since $p^{-1} \in \theta \backslash \theta_v$; therefore $\theta_{\dot{v}} \subseteq \theta$. We conclude that $H_E \subseteq \bigcap_{\theta \in \mathbb{B}} \theta = \mathbb{Q} \cdot R$, hence (3_E) holds. Moreover, it follows that the set $\{\theta_{\dot{v}} \mid v \in \vartheta\} = \{\theta_{\mathscr{P}} \mid \mathscr{P} \in \mathscr{S}_E\}$ is saturated; therefore (2_E) is true. $\square$

3. The p-adic Kochen ring $R_E(K)$

For arbitrary integers $e \geq 1$ and $f \geq 1$, Prestel and Roquette introduced in [4, Section 6] the *p-adic Kochen operator of type* (e,f) as

$$\gamma_{e,f}(X) = \frac{1}{p} \cdot ((X^{p^f} - X) - (X^{p^f} - X)^{-1})^{-e}$$

and, for any field K of characteristic zero, the *p-adic Kochen ring of type* (e,f) as the ring $R_{e,f} = R_{e,f}(K)$ of quotients of the form $a/(1 + p \cdot b)$, with $a, b \in \mathbb{Z}[\gamma_{e,f}(K) \backslash \{\infty\}]$ and $1 + p \cdot b \neq 0$.

This ring admits K as its field of quotients and is related to the p-valuations of K in the following way. Let $\vartheta_{e,f} = \vartheta_{e,f}(K)$ be the set of all p-valuations of K such that $e_v \leq e$ and f_v divides f; then we have (cf. [4, 6.8 and 6.9]):

3.1. *For any valuation v of K we have $v \in \vartheta_{e,f}$ if and only if $R_{e,f} \subseteq \theta_v$ and $vp > 0$.*

3.2. *The integral closure of $R_{e,f}$ is equal to the intersection $\bigcap_{v \in \vartheta_{e,f}} \theta_v$, which is a Bezout (and therefore a Prüfer) ring of K.*

In the case $e = 1$, even the following equality holds (cf. [4, 6.14]):

3.3. $R_{1,f} = \bigcap_{v \in \vartheta_{1,f}} \theta_v$.

Since $\vartheta_{1,f}$ is equal to the set of all p-valuations of type $E = \mathbb{Q}_{p,f}$, it follows easily[2] the following corollary.

Corollary 3.4. *In the case $E = \mathbb{Q}_{p,f}$, if K is formally p-adic of type E, then $\vartheta_{1,f}$ is saturated and therefore satisfies the hypothesis of Theorem 2.2.*

In this way, by the use of the Kochen ring $R_{1,f}(K)$, it was shown in [1] that, in the case $E = Q_{p,f}$, for any field K which is formally p-adic of type E, the holomorphy ring $H_E(K)$ has the desired properties (2_E) and (3_E). On the other hand, in the general case, it was not possible to use of the Kochen ring $R_{e,f}(K)$, since, in general, there exists $v \in \vartheta_{e,f}$ which is not of type E (see [3]).

In order to deal with the general case, Roquette [5] introduced, for an arbitrary finite extension E of $\mathbb{Q}_p$, the notion of a Kochen ring of type E as a certain over-ring of $R_{e,f}(K)$. For its definition, the following facts are crucial.

[2] Actually, (3.3) is not needed for the proof of Corollary 3.4 (see the proof of Corollary 3.12).

Lemma 3.5. *For given integers $e \geq 1$ and $f \geq 1$, there exist only finitely many extensions E of $\mathbb{Q}_p$ (within an algebraic closure of $\mathbb{Q}_p$) such that $e_E = e$ and $f_E = f$.*

Proof. See [3, p. 236] and the literature cited there. $\square$

Lemma 3.6. *$\mathcal{O}_E = \mathbb{Z}_p[\beta]$ for some algebraic integer β.*

Proof. We have $\mathcal{O}_E = \mathbb{Z}_p[\alpha]$ for some $\alpha \in \mathcal{O}_E$, and $\mathcal{O}_E = \mathbb{Z}_p[\beta]$ for any $\beta \in \mathcal{O}_E$ such that $v_E(\beta - \alpha) \leq 2$ (cf. [7, 3–3]). By the continuity of polynomial roots, any monic polynomial $P \in \mathbb{Z}_p[X]$ which is sufficiently proximate to the minimal polynomial of α over $\mathbb{Q}_p$ has a root β of this kind, and from Krasner's lemma it follows that P is irreducible (cf. [2, Section 24]). In particular, P may be chosen in $\mathbb{Z}[X]$; then β is integral over $\mathbb{Z}$. $\square$

Let E be a finite extension of $\mathbb{Q}_p$. We choose (e, f) such that $e_E \leq e$ and f_E divides f, and we denote by $E_1, \ldots, E_r$ those finite extensions of $\mathbb{Q}_p$ for which $e_{E_i} \leq e, f_{E_i}$ divides $f (i = 1, \ldots, r)$ and which cannot be imbedded in E. Finally, for $i = 1, \ldots, r$ we choose an algebraic integer β_i such that $\mathcal{O}_{E_i} = \mathbb{Z}_p[\beta_i]$ and denote by F the product $P_1 \ldots P_r \in \mathbb{Z}[X]$ of the minimal polynomials P_i of β_i over $\mathbb{Q}$. In the case $r = 0$ (which occurs if and only if $e_E = e = 1$ and $f_E = f$) we set $F = 1$.

Lemma 3.7. *There is a $k \in \mathbb{N}$ such that $v_E(F(a)) \leq k \cdot v_E p$ for all $a \in E$.*

Proof. Since $\mathcal{O}_E$ is compact, there would otherwise be a sequence in $\mathcal{O}_E$ which converges to a zero α of F, that is, a zero of P_i for some $i \in \{1, \ldots, r\}$. Therefore $\beta_i \to \alpha$ would define an imbedding of $E_i = \mathbb{Q}_p[\beta_i]$ into E, which is impossible. $\square$

Choosing an integer $k \in \mathbb{N}$ which satisfies Lemma 3.7, we consider the operator

$$\delta_E(X) = \frac{p^k}{F(X)}$$

and, for any field K, we denote by $\delta_E K$ the set of elements $\delta_E(a)$ such that $a \in K$ and $F(a) \neq 0$. By means of $\delta_E K$, the p-valuations of type E can be characterized within $\vartheta_{e,f}$.

Proposition 3.8. *Let $v \in \vartheta_{e,f}$; then v is of type E if and only if $\delta_E K \subseteq \mathcal{O}_v$.*

Proof. Let κ be the canonical homomorphism from $\mathcal{O}_{\hat{v}}$ onto $K_{\hat{v}}$.

(a) Let v be of type E; so we may assume $K_{\hat{v}} \subseteq E$. For any $a \in \mathcal{O}_{\hat{v}}$ we have $F(a) \in \mathcal{O}_{\hat{v}}$ and, by Lemma 3.7, $v(F(a)) = v_E(F(\kappa a)) \leq k \cdot v_E p = k \cdot vp$; therefore $v(\delta_E(a)) \geq 0$.

Actually, this inequality holds for all $a \in K$; in fact, if $a \notin \theta_{\mathring{v}}$ then $va < 0$, $v(F(a)) < 0$ and therefore $v(\delta_E(a)) = k \cdot vp - v(F(a)) > 0$.

(b) Assume that v is not of type E; then the completion of $(K_{\mathring{v}}, \mathring{v})$ cannot be imbedded in (E, v_E) and is therefore of the form (E_j, v_{E_j}) for some $j \in \{1, \ldots, r\}$. Since β_j is a root of F, we have $v_{E_j}(F(\alpha)) > k \cdot v_{E_j}(p)$ for all $\alpha \in E_j$ which are sufficiently proximate to β_j. Since $K_{\mathring{v}}$ is dense in E_j, we can choose $\alpha = \kappa a$ for some $a \in \theta_{\mathring{v}}$ with $F(\alpha) \neq 0$. We conclude that $v(F(a)) = \mathring{v}(F(\alpha)) > k \cdot vp$ and $F(a) \neq 0$; therefore $\delta_E(a) \in \delta_E K$ and $v(\delta_E(a)) < 0$. □

In order to characterize the p-valuations of type E within the set of *all* valuations of K, we need the *p-adic Kochen ring of type E*, denoted by $R_E = R_E(K)$ and defined as the ring of all quotients of the form $a/(1 + p \cdot b)$, with $a, b \in R_{e,f}[\delta_E K]$ and $1 + p \cdot b \neq 0$. Note that this ring depends not only on E but also on the choice of e, f, $\beta_1, \ldots, \beta_r$ and k, and that $R_E = R_{e,f}$ in the case $r = 0$, $e = e_E$, $f = f_E$.

Let $\vartheta_E = \vartheta_E(K)$ be the set of all p-valuations of K of type E; obviously $\vartheta_E \subseteq \vartheta_{e,f}$. For the set ϑ_E the following analogue of 3.1 holds.

Theorem 3.9. *For any valuation v of K we have $v \in \vartheta_E$ if and only if $R_E \subseteq \theta_v$ and $vp > 0$.*

Proof. If $v \in \vartheta_E$, then $v \in \vartheta_{e,f}$, hence $R_{e,f} \subseteq \theta_v$ and $vp > 0$ by 3.1; from Proposition 3.8 it follows that $R_E \subseteq \theta_v$. On the other hand, if $R_E \subseteq \theta_v$ and $vp > 0$, then $R_{e,f} \subseteq \theta_v$, hence $v \in \vartheta_{e,f}$, by 3.1. Suppose that $v \notin \vartheta_E$; then $\delta_E K \nsubseteq \theta_v$ by Proposition 3.8, contradicting $R_E \subseteq \theta_v$. □

In the following corollary it can be considered as an analogue of [4, 6.7 and 6.8]. It characterizes those prime ideals of R_E which occur as a center of some $v \in \vartheta_E$.

Corollary 3.10. *K is formally p-adic of type E (i.e. ϑ_E is nonempty) if and only if $R_E \neq K$. In this case, for any prime ideal $\mathfrak{G}$ of R_E, the following conditions are equivalent:*

(i) *$\mathfrak{G}$ is a maximal ideal of R_E;*

(ii) *$p \in \mathfrak{G}$;*

(iii) *$\mathfrak{G} = \mathcal{M}_v \cap R_E$ for some $v \in \vartheta_E$;*

in particular, p is contained in the Jacobson radical of R_E.

Proof. If ϑ_E is nonempty, say $v \in \vartheta_E$, then by Theorem 3.9, p is a nonunit in θ_v and therefore also in the subring R_E; hence $R_E \neq K$.

Assume that $R_E \neq K$. We claim that p is a nonunit in $S = R_{e,f}[\delta_E K]$. In fact; suppose p is a unit, and let $s \in S$ be arbitrary. Then $s/p - 1/p = d_S$ when $d_S \in S$, and hence $s = 1 + pd_S$; it follows that the quotient field of S is contained in R_E, a contradiction.

(i) $\Rightarrow$ (ii): Since p is a nonunit in S, we have $1 + p\cdot c \neq 0$ for all $c \in S$. Hence $1 + pc \in R_E^*$ by the definition of R_E, and therefore $1 + pc \notin \mathfrak{G}$. Suppose that $p \notin \mathfrak{G}$; then $p\cdot z \equiv 1 \bmod \mathfrak{G}$ for some $z \in R_E$, say $z = a/(1 + p\cdot b)$ with $a, b \in S$; therefore $1 + p\cdot(b - a) \in \mathfrak{G}$, a contradiction.

(ii) $\Rightarrow$ (iii): By the extension theorem (cf. [2, (9.7)]), there is a valuation v of K such that $R_E \subseteq \mathcal{O}_v$ and $\mathfrak{G} = \mathcal{M}_v \cap R_E$; therefore $v \in \vartheta_E$, by Theorem 3.9.

(iii) $\Rightarrow$ (i): Since $R_E/\mathfrak{G}$ is contained in the finite field K_v, $R_E/\mathfrak{G}$ is a field, too, and therefore $\mathfrak{G}$ is a maximal ideal of R_E.

Finally, from the existence of maximal ideals we conclude that ϑ_E is nonempty. $\square$

It is well known that the integral closure R' of any subring R of K is equal to the intersection $\bigcap_{\mathcal{O}\in\mathbb{B}} \mathcal{O}$, where $\mathbb{B}$ is the set of valuation rings $\mathcal{O}$ of K such that $R \subseteq \mathcal{O}$ and $\mathcal{M} \cap R$ is a maximal ideal of R (cf. [2, (10.8)]). From Theorem 3.9, Corollary 3.10 and from 3.2 and the inclusion $R_{e,f} \subseteq R_E$ we obtain the following corollary.

Corollary 3.11. *The integral closure R_E' of R_E is equal to the intersection $\bigcap_{v\in\vartheta_E} \mathcal{O}_v$, which is a Bezout (and therefore a Prüfer) ring of K.*

We are now able to generalize Corollary 3.4 to an arbitrary finite extension E of $\mathbb{Q}_p$.

Corollary 3.12. *If K is formally p-adic of type E, then $\vartheta_E(K)$ is saturated and therefore satisfies the hypothesis of Theorem 2.2.*

Proof. Let $\mathcal{O}$ be a valuation ring of K which contains the ring $S = \bigcap_{v\in\vartheta_E} \mathcal{O}_v$, and let $\mathcal{M}$ be a maximal ideal of S which contains $\mathfrak{G} = \mathcal{M}_\mathcal{O} \cap S$; then $S_\mathcal{M} = \mathcal{O}_{v_1}$ for some valuation v_1 of K with $S \subseteq \mathcal{O}_{v_1} \subseteq \mathcal{O}$ and $\mathcal{M}_{v_1} \cap S = \mathcal{M}$ (cf. [2, Section 11]). Since $S = R_E'$ by Corollary 3.11, $\mathcal{M}_{v_1} \cap R_E = \mathcal{M} \cap R_E$ is a maximal ideal of R_E by Corollary 3.10, hence $v_1 p > 0$. From Theorem 3.9 we conclude that $v_1 \in \vartheta_E$. Therefore, ϑ_E is saturated. $\square$

This corollary allows us to complete Section 3 by the following theorem.

Theorem 3.13. *If K is formally p-adic of type E, then $H_E(K)$ satisfies (1_E), (2_E), (3_E), (4_E) and, in particular, is equal to $\mathbb{Q}\cdot R_E'(K)$.*

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References

[1] C.M.M. Dutra, O anel de holomorfia p-ádico, Inform. Mat. Sér. F, IMPA-RJ (1985).

[2] O. Endler, Valuation Theory, Universitext (Springer, Berlin, 1972).

[3] W. Narkiewich, Elementary and Analytic Theory of Algebraic Numbers (Polish Scientific Publishers, Warszawa, 1974).

[4] A. Prestel and P. Roquette, Formally p-adic fields, Lecture Notes in Mathematics, Vol. 1050 (Springer, Berlin, 1984).

[5] P. Roquette, Kochen-Ring R_E, unpublished (letter to O. Endler, 1985).

[6] H.W. Schülting, Über die Erzeugendenanzahl invertierbarer Ideale in Prüferringen, Comm. Algebra 7 (1979) 1331–1349.

[7] E. Weiss, Algebraic Number Theory (McGraw-Hill, New York, 1963).